On Moduli Spaces of Triangles

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Title: On Moduli Spaces of Triangles

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Abstract

What is the shape of the set of similarity classes of triangles? In this senior project, we explore and answer this question using moduli space theory. Euclid's classical triangle similarity theorems state that two triangles are similar if and only if they have the same ordered interior angles or ordered ratios of side lengths. These two "angle-angle-angle" and "side-side-side" similarity theorems are equivalent on nondegenerate triangles. However, if we consider a sequence of triangles that flattens out and approaches triangle with zero area, these theorems cease to be equivalent. The "angle-angle-angle" theorem gives rise to a torus, while the "side-side-side" theorem results in a sphere. Since these two spaces are not homeomorphic, neither can be the moduli space of similarity classes of triangles. In this project, we provide explicit constructions of both the sphere and the torus as spaces of triangles. We unify these two spaces by considering both types of similarity in our preprint [BGGL24], where we prove that *Dyck's surface*, the connected sum of three real projective planes, is the a fine moduli space of labeled, oriented, possibly-degenerate similarity classes of triangles.

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1. Introduction

What is the shape of the set of similarity classes of triangles?

We explore and answer this question through the use of moduli space theory in this senior project.

This report is organized as follows. First, we provide some motivation for why people like to study moduli spaces. In Section 2, we explore the space obtained when considering similarity classes of triangles as triples of angles (α, β, γ) such that $\alpha + \beta + \gamma = \pi$. In Section 3, we examine another notion of similarity by explore the space obtained when instead considering triangles by their vertices $(A, B, C) \in \mathbb{C}^3$ and then imposing an equivalence relation on them. This is the well-known "shape sphere," and we provide details on the construction as well as a variety of observations.

In our followup preprint [BGGL24], we explore the rigorous categorical definition of moduli spaces and stacks and prove that Dyck's surface, the connected sum of three real projective planes, is the fine moduli space of labeled, oriented possibly-degenerate similarity classes of triangles. We also prove that Dyck's surface is a smooth manifold.

1.1. **Why moduli spaces?**

Before discussing the more technical notion of moduli spaces, we begin with a discussion of parameter spaces. A *parameter space* is a space that classifies a given collection of objects that one is interested in.

We consider a familiar example. Let $\mathcal{E}$ be the set of all points on the surface of Earth. Thne the sphere S^2 is a parameter space for $\mathcal{E}$, since points on Earth are in clear one-to-one correspondence with points on the sphere. Furthermore, points that are "close" on Earth are "close" on the sphere, so there is a notion of continuity. We also know that the sphere is the "best" object to classify points on Earth. In particular, any two-dimensional map projection must distort areas on Earth somehow, and they all descend from the sphere. In a sense, the sphere parameterizes points on Earth "universally so." Armed with this sphere, we can answer all sorts of interesting questions about our underlying set $\mathcal{E}$, such as

What is the probability that a random point on the surface of Earth is in the ocean?

By analogy, finding a parameter space for a more abstract set than $\mathcal{E}$ requires mathematical cartography, charting out an abstract world.

1.2. **Similarity Classes of Triangles.**

The "abstract world" we choose to explore is the space of all Euclidean plane triangles up to similarity.

Two Euclidean plane triangles are similar if one can be scaled, rotated, and/or translated to agree with the other. Finding a moduli space for similarity classes of triangles is a frequent first exercise for those learning moduli space theory, and so we are certainly not the first to explore this topic.

Euclid proved that two triangles are similar if and only if they have the same interior angles in the same order [Euc56]. That is known as "angle-angle-angle" similarity. Similarly, his "side-side-side" similarity theorem states that two triangles are similar if and only if they have the same ratios of side lengths. Critically, these theorems are equivalent only on *nondegenerate* triangles, triangles with nonzero area, triangles whose three vertices are not collinear. Some examples of degenerate triangles are illustrated in Figure 1.1.

Figure 1.1. Example multiplicity 1, 2, and 3 degenerate triangles

Why would anybody care about "flat triangles," triangles with zero area? These degenerate triangles are the limit points of sequences of nondegenerate triangles, and so if we want to have any hope of our moduli space being closed in the metric topology, then we must consider these limit points.

It turns out that the angle-angle-angle and side-side-side theorems are no longer equivalent on degenerate triangles. Why? Given a degenerate triangle where exactly two vertices coincide, there is no canonical choice for the interior angles at the coinciding vertices. Similarly, a degenerate triangle with three distinct collinear vertices will have interior angles $0, 0, \pi$ in some order, but these angles do not uniquely determine the ratios of the side lengths of the triangle.

This nonequivalence of these theorems is manifested in the different topologies of the parameter spaces that they induce. Under angle-angle-angle similarity, we get a torus of triangles. Side-side-side similarity gives rise to a sphere of triangles. These are clearly distinct spaces, and so neither can be the moduli space of similarity classes of triangles, for they must parameterize these families "universally so."

The torus is missing degenerate triangles consisting of three distinct collinear points, while the sphere is missing degenerate triangles where exactly two vertices coincide. In [BGGL24], we blowup the two spaces at the points where these degenerate triangles are missing. It is a result of Walther von Dyck that the blow up the torus at one point is homeomorphic to the blowup of a sphere at three points [Dyc88]. We show that *Dyck's surface*, the result of these blowups, is a smooth manifold and a find moduli space for the set of similarity classes of triangles.

2. The Torus of Triangles

2.1. **Introduction.** The study of triangles in the plane dates back to Euclid and is one of the oldest and most thoroughly investigated subjects in all of mathematics.[1] Straightedge-compass constructions, connections with Apollonian problems, and the Euclidean geometry of the plane have been studied exhaustively. It is a fascinating, elemental subject. For a glimpse at the area's extent, the reader is invited to browse a list of thousands of geometrically defined triangle "centers", see [Kim98].

For our purposes, a *moduli space* is a space of points that represent members of a set of mathematical objects, and whose geometry has a natural relevance to the objects' characteristic structures. Thus the moduli space is useful for studying the families of these objects as a whole, to make statistical calculations, or just to visualize one family in the context of others.

Our paper was motivated by the question of what happens near the boundary of a commonly drawn moduli space of triangles called the *triangle of triangles*, see e.g. [ES15, Figure 2]. In our study of continuous families of triangles – ones parameterized by continua of real numbers – we wondered what would happen to a family that "breached the border" of this space, a border that appears not to consist of triangles at all, but rather of degenerate, "flat" triangles. As the renowned number theorist Barry Mazur put it, *it is precisely in the neighborhoods of such regions in many of the moduli spaces currently studied where profound things take place.* ([Maz18, p.7]). Though it may seem unlikely that anything profound should occur in such an elementary example (see [Maz18, p.7]), we show how the space of similarity classes of triangles (the triangle of triangles) extends naturally to the space of similarity classes of *oriented* triangles, to form an abelian Lie group: a topological torus of similarity classes of labeled, oriented triangles. We call it, of course, the *torus of triangles.*

The problem of constructing a moduli space of triangles is frequently posed as an elementary exercise that demonstrates the properties and challenges of more complex spaces. The idea is suggested by Lewis Carroll, who in "Pillow Problem 58" asked for the probability that a randomly chosen triangle is obtuse (see [Dod58]). As pointed out in [CNSS19], the first mention of it in the literature appears to be in *The Lady's and Gentleman's Diary* [Woo61], where W.S.B. Woolhouse asked the following: "In a given circle a regular polygon is inscribed, and lines are drawn from each of its angles to the center of the circle. Required the ratio of the number of the triangles which are acute-angled to that of those which are obtuse-angled." This question has intriguingly different answers, depending on the construction of the space.

Literature. There have been many constructions of moduli spaces of triangles, including [CNSS19], [ES15], [Guy93], [Ken85], [Por94].

The article [Por94] makes a convincing case that a reasonable moduli space should admit a transitive action by a compact group. The points are thereby assigned equal priority, and the different regions can be assigned finite measures since the group is compact. Portnoy further suggests that the "right" distribution should align with the spherically symmetrical construction $\mathbb{P}^5(\mathbb{R})$ using the six coordinates of the triangle's three vertices (up to scalar multiplication). This idea is championed in [ES15], see below.

[1] The contents of this section is the preprint that I coauthored with Eric Brussel [BG23]. It has been reproduced here with minor modifications.

In [CNSS19] a space with a transitive action by a compact group is constructed that generalizes to n-gons and introduces valuable pedagogical techniques. The paper succeeds in obtaining a uniform distribution of triangles, but at the expense of distorting the relative measures of triangles in order to make the plane conform to a sphere. This has the effect of over-valuing short side lengths, hence obtuse triangles, and indeed their computation of the obtuse-to-acute ratio is much higher than the 3-to-1 value advocated by Portnoy and many others.

Since our construction actually *is* a compact Lie group, it admits a transitive action by a compact group, and so passes Portnoy's test. Furthermore, the measurements we make agree with those of [Por94] based on the $\mathbb{P}^5$ example, although our measure appears to be quite different, as noted in [ES15, 1.1].

In the excellent treatment [ES15], which aims to rejuvenate the study of shape theory, the moduli space is constructed as in [Por94], and the focus is on the normal distribution on the six coordinates of three vertices, which they apply to obtain a probability $P(O) = 3/4$ for a random triangle to be obtuse. They produce our angle-based moduli space in passing, and apply the uniform angle distribution to obtain the same answer $P(O) = 3/4$ for a random triangle to be obtuse. They point out how curious it is that these two spaces give the same result, in spite of the fact that they come from fundamentally different measures.

2.2. The Triangle of Triangles.

Definition 2.2.1. (a) A *triangle* is a plane figure consisting of three vertices and three straight edges connecting them. At each vertex is a positive *interior angle*, which is the angle between incident edges. If traversed counterclockwise around the vertex, the value of an interior angle is between 0 and π, inclusive; otherwise it is between $-\pi$ and 0. If all angles are traversed counterclockwise they sum to π, otherwise $-\pi$.

(b) A triangle is *degenerate* if its vertices are colinear, in which case at least one interior angle is zero, and *nondegenerate* otherwise, in which case all interior angles are convex.

(c) Two nondegenerate triangles are *similar* if the absolute values of their interior angles are equal, in some order.

(d) A *labeled triangle* is a triangle together with a labeling A, B, C of its vertices.

(e) A nondegenerate labeled triangle has *positive orientation* if its labeling is lexicographic when the vertices are traversed counterclockwise, and *negative orientation* if it is anti-lexicographic when traversed counterclockwise. A degenerate triangle has *zero orientation*.

(f) **Notation**. We write $\triangle ABC$ for the triangle with vertices $A, B, C \in \mathbb{R}^2$, $[\triangle ABC]$ for its labeled, oriented similarity class, and $\triangle[\theta_1, \theta_2, \theta_3]$ for the labeled, oriented similarity class with angles $\theta_1, \theta_2, \theta_3$ assigned to the ordered vertices A, B, C. We write $|[\triangle ABC]|$ and $|\triangle[\theta_1, \theta_2, \theta_3]|$ for the *absolute* (unoriented, unlabeled) similarity class of the triangle $\triangle ABC$.

Remark 2.2.2. A scalene triangle $\triangle ABC$ has a single absolute similarity class, but twelve labeled, oriented similarity classes: six corresponding to the six ways of assigning the three angles assigned to the three vertices, and two for each orientation, corresponding to whether the labeled vertices are in lexicographic or anti-lexicographic order when traversed counterclockwise in the plane.

2.2.3. *Triangle of Triangles.* We next use interior angles to define a pair of triangular planar regions that parameterize the similarity classes of labeled, oriented triangles, and provide a uniform metric with which we will compute relative measures of different triangle types. In Section 2.4, we will show that they naturally glue together on their degenerate boundaries to form an abelian Lie group homeomorphic to a torus, called the *Clifford torus*. The fact that it is a group solves the uniformity problem raised in [Por94], and addressed in [CNSS19] for labeled, oriented similarity classes.

Definition 2.2.4. The *triangle of triangles* and the *shadow triangle of triangles* are the planar regions

$$\mathcal{T}_+ = \{(\alpha,\beta,\gamma) : \alpha+\beta+\gamma = \pi,\ 0 \le \alpha,\beta,\gamma \le \pi\}$$
$$\mathcal{T}_- = \{(\alpha,\beta,\gamma) : \alpha+\beta+\gamma = -\pi,\ -\pi \le \alpha,\beta,\gamma \le 0\}$$

The interiors $\mathcal{T}_+^\circ$ and $\mathcal{T}_-^\circ$ are the points satisfying $\alpha\beta\gamma \neq 0$, representing the similarity classes of nondegenerate labeled, oriented triangles. The borders $\partial\mathcal{T}_+$ and $\partial\mathcal{T}_-$ are the points satisfying $\alpha\beta\gamma = 0$, representing degenerate labeled triangles (see Figure 2.1), for which similarity has yet to be defined. Since α,β,γ are interior angles, $\triangle[\alpha,\beta,\gamma]$ and $-\triangle[\alpha,\beta,\gamma] := \triangle[-\alpha,-\beta,-\gamma]$ have the same absolute measure at the vertices A, B, C, but when drawn on the plane those vertices are in opposite orders when traversed counterclockwise.

2.2.5. *Taxonomy of Triangles.* The six main types of triangles are *equilateral*, *isosceles*, *scalene*, *right*, *acute*, and *obtuse*. In the triangles of triangles in Figure 2.1 the three altitudes represent isosceles triangles, and the darker inscribed triangular regions represent acute triangles. Thus in the set of similarity classes of nondegenerate labeled, oriented triangles we find:

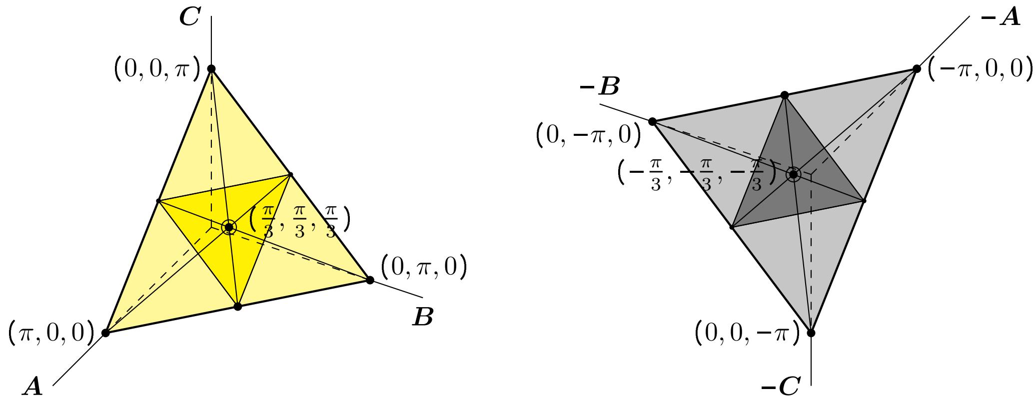
(a) Two nondegenerate equilateral triangles $\pm\triangle[\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}]$.
(b) Nondegenerate isosceles triangles $\triangle[\alpha,\alpha,\gamma]$, $\triangle[\alpha,\beta,\alpha]$, or $\triangle[\alpha,\beta,\beta]$.
(c) Nondegenerate right triangles $\triangle[\alpha,\beta,\gamma]$ with either α,β, or γ equal to $\pm\frac{\pi}{2}$.
(d) Nondegenerate scalene triangles $\triangle[\alpha,\beta,\gamma]$ with α,β,γ all distinct.

We extend these classifications to degenerate triangles as follows.

Definition 2.2.6. We say a degenerate element $\triangle = \triangle[\alpha,\beta,\gamma]$, which by definition has at least one zero interior angle, is

(a) *equilateral* if $\triangle \in \{\triangle[\pm\pi,0,0],\ \triangle[0,\pm\pi,0],\ \triangle[0,0,\pm\pi]\}$;
(b) *isosceles* if $\triangle$ is equilateral or $\triangle \in \left\{\triangle[0,\pm\frac{\pi}{2},\pm\frac{\pi}{2}],\ \triangle[\pm\frac{\pi}{2},0,\pm\frac{\pi}{2}],\ \triangle[\pm\frac{\pi}{2},\pm\frac{\pi}{2},0]\right\}$;
(c) *right* if isosceles;
(d) *scalene* if $\triangle \in \left\{\triangle[0,\beta,\gamma],\ \triangle[\alpha,0,\gamma],\ \triangle[\alpha,\beta,0] : \alpha,\beta,\gamma \neq 0,\pm\frac{\pi}{2},\pm\pi\right\}$.

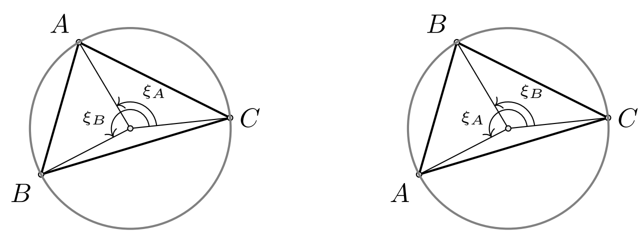
2.3. **The Torus of Relative Arguments.**



(a) Triangle of Triangles $\mathcal{T}_+$ (b) Shadow Triangle of Triangles $\mathcal{T}_-$

Figure 2.1. The Triangles of Triangles

In this section we construct another space whose general points represent similarity classes of nondegenerate labeled, oriented triangles, and whose continuity leads to a natural definition of similarity for degenerate labeled triangles, which form the border regions $\partial\mathcal{T}_+$ and $\partial\mathcal{T}_-$ above, and with which we will glue $\mathcal{T}_+$ and $\mathcal{T}_-$ together. The idea is based on the observation that a nondegenerate labeled, oriented similarity class $[\triangle ABC]$ is completely determined by the two ordered arguments ξ_A and ξ_B relative to C. The triangle can then be inscribed in the unit circle, as in Figure 2.2, where we write the arguments with representatives between 0 and 2π.



Positive Orientation Negative Orientation

Figure 2.2. Relative Arguments and Orientation

2.3.1. *Torus of Relative Arguments.*

Construction. Let S^1 denote the circle Lie group, $\mathbb{R}^1 = (\mathbb{R}^1, +)$ its Lie algebra, and $\exp : \mathbb{R}^1 \longrightarrow \mathsf{S}^1$ the exponential $\xi \mapsto e^{i\xi}$, which is a homomorphism. We call ξ an *argument*. The $\mathbb{R}$-linear homomorphism $\delta_+ : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $\delta_+(\xi_1, \xi_2, \xi_2) = (\xi_1 - \xi_3, \xi_2 - \xi_3)$ determines a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^1 & \xrightarrow{\Delta} & \mathbb{R}^3 & \xrightarrow{\delta_+} & \mathbb{R}^2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \\ 1 & \longrightarrow & \mathsf{S}^1 & \xrightarrow{\Delta} & (\mathsf{S}^1)^3 & \xrightarrow{\delta} & \mathcal{T} & \longrightarrow & 1 \end{array}$$

where Δ is the diagonal embedding, and $\mathcal{T} \simeq (\mathsf{S}^1)^2$ is a torus. We call $\mathbb{R}^2$ the *plane of relative arguments*. The maps δ_+ and δ are split by the $\mathbb{R}$-linear map $\sigma_+ : (\xi_1, \xi_2) \mapsto (\xi_1, \xi_2, 0)$ and the homomorphism $\sigma : (e^{i\xi_1}, e^{i\xi_2}) \mapsto (e^{i\xi_1}, e^{i\xi_2}, 1)$, respectively. Since left multiplication by S^1 is the same as the (left) rotation action by $\mathrm{SO}(2)$, we have

$$\mathcal{T} = (\mathsf{S}^1)^3/\mathrm{SO}(2)$$

On the other hand, since the kernel of exp consists of the lattice $\{(2k\pi, 2\ell\pi) : k, \ell \in \mathbb{Z}\}$, $\mathcal{T} \simeq \mathbb{R}^2/2\pi\mathbb{Z}^2$, which is the usual definition of the Clifford torus.

Definition 2.3.2. (a) The *torus of relative arguments* is the abelian torus group $\mathcal{T}$ constructed above, with points interpreted as ordered triples of points on S^1 up to rotations, each represented by a unique ordered triple $(e^{i\xi_1}, e^{i\xi_2}, 1)$, and with pointwise product

$$(e^{i\xi_1}, e^{i\xi_2}, 1) \cdot (e^{i\xi_1'}, e^{i\xi_2'}, 1) = (e^{i(\xi_1+\xi_1')}, e^{i(\xi_2+\xi_2')}, 1).$$

(b) A triple $(e^{i\xi_1}, e^{i\xi_2}, 1)$ is *nondegenerate* if the three points are distinct, and *degenerate* otherwise. Thus a degenerate triple satisfies $\xi_1 = 2k\pi$, $\xi_2 = 2k\pi$, or $\xi_1 - \xi_2 = 2k\pi$ for $k \in \mathbb{Z}$.

(c) A nondegenerate triple has *positive orientation* if the forward (left-to-right) reading of coordinates in the triple goes counterclockwise around S^1, and *negative orientation* otherwise. A degenerate triple has *zero orientation*.

2.3.3. *Connection of $\mathcal{T}$ to Triangles.* Each point (ξ_1, ξ_2) in the plane of relative arguments maps to a point $(e^{i\xi_1}, e^{i\xi_2}, 1)$ on $\mathcal{T}$, corresponding to a unique similarity class of labeled, oriented triangle. We illustrate the plane in Figure 2.3, with a shaded fundamental domain $0 \le \xi_1, \xi_2 < 2\pi$, and solid lines representing the preimage of the degenerate triangles on $\mathcal{T}$. The fundamental domain is divided into two parts according to the orientations of the corresponding triangles.

The construction of a borderless parameter space $\mathcal{T}$ arising from Figure 2.3 depends on the incorporation of orientation. In this space, a continuous family *changes orientation* when it crosses the degenerate border. In moduli spaces (such as $\mathcal{T}_+$) that do not take orientation into account, a path that intersects the border transversally has a singularity, and must reverse course to stay in the space. By adding orientation we obtain what we feel is a more natural space. We will show later that the usual space of (unoriented, unlabeled) similarity classes is a topological quotient.

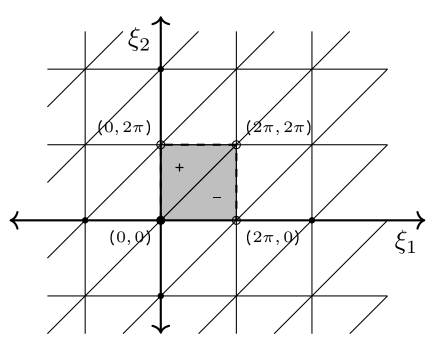


Figure 2.3. The Plane of Relative Arguments

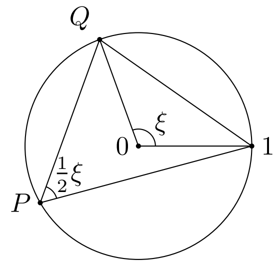


Figure 2.4. Euclid's Elements Book III, Proposition 20

2.4. **The Torus of Triangles.** In this section we make the correspondence between points on the torus and labeled oriented similarity classes more explicit by defining a map $\mathcal{T}_+ \sqcup \mathcal{T}_- \longrightarrow \mathcal{T}$ from the triangles of triangles of Definition 2.2.4 to the torus of Definition 2.3.2 giving us an explicit interior angle description of the triangles represented by points of $\mathcal{T}$, and inducing a definition of similarity for degenerate triangles. This also serves the purpose of imposing on $\mathcal{T}$ the standard uniform measure implicit in $\mathcal{T}_+ \sqcup \mathcal{T}_-$.

2.4.1. *Map from Triangles of Triangles to Torus of Relative Arguments.*

Construction. A point on $[\triangle ABC] = \triangle[\alpha,\beta,\gamma] \in \mathcal{T}_+$ satisfies $\alpha,\beta,\gamma \in [0,\pi]$ and $\alpha+\beta+\gamma = \pi$, and $\triangle[\alpha,\beta,\gamma] \in \mathcal{T}_-$ satisfies $\alpha,\beta,\gamma \in [-\pi,0]$ and $\alpha+\beta+\gamma = -\pi$. Proposition 20 of Euclid's Book III ([Euc56], see also [Joy98]) states, *in a circle the angle at the center is double the angle at the circumference when the angles have the same circumference as base.* Therefore if we inscribe the similarity class of a nondegenerate triangle $\triangle[\alpha,\beta,\gamma]$ in S^1 with third vertex $C = 1$, we can compute the arguments of A and B from α and β, and so define a map from the triangles of triangles to the torus of relative arguments. To wit, if the triangle is positively oriented then $\alpha,\beta,\gamma > 0$, and then $P = B$, $Q = A$, and $\beta = \frac{1}{2}\xi > 0$ in Figure 2.4. Since ξ is the counterclockwise argument of A, $A = e^{i2\beta}$, and similarly $B = e^{i(2\pi-2\alpha)} = e^{-i2\alpha}$. The resulting triple $(e^{i2\beta}, e^{-i2\alpha}, 1)$ is positively oriented, since the left-to-right reading goes counterclockwise around the circle. If the triangle is negatively oriented then $\alpha,\beta,\gamma < 0$, and $P = A$, $Q = B$, and $\alpha = -\frac{1}{2}\xi < 0$. Then $B = e^{-i2\alpha}$, and similarly $A = e^{-i(2\pi-2\beta)} = e^{i2\beta}$. This time the resulting triple $(e^{i2\beta}, e^{-i2\alpha}, 1)$ is negatively oriented. This motivates the following definition.

Definition 2.4.2. Suppose $\triangle = \triangle[\alpha,\beta,\gamma] \in \mathcal{T}_+ \sqcup \mathcal{T}_-$, with $\alpha,\beta,\gamma \in [0,\pi]$ if $\triangle \in \mathcal{T}_+$, and $\alpha,\beta,\gamma \in [-\pi,0]$ if $\triangle \in \mathcal{T}_-$. Define

$$\rho : \mathcal{T}_+ \sqcup \mathcal{T}_- \longrightarrow \mathcal{T}$$
$$\triangle[\alpha,\beta,\gamma] \longmapsto (e^{i2\beta}, e^{-i2\alpha}, 1)$$

Theorem 2.4.3. *The map ρ is surjective, preserves orientation, and inscribes $\triangle[\alpha,\beta,\gamma] \in \mathcal{T}_+ \sqcup \mathcal{T}_-$ in S^1 as the triple $(e^{i2\beta}, e^{-i2\alpha}, 1)$.*

Let $\partial(\partial(\mathcal{T}_+ \sqcup \mathcal{T}_-))$ denote the six vertices of the border $\partial(\mathcal{T}_+ \sqcup \mathcal{T}_-)$, and let $(\partial\mathcal{T}_+)^\circ \sqcup (\partial\mathcal{T}_-)^\circ$ denote the interior of the border. Assume $0 \le \alpha,\beta \le \pi$. Then the degenerate points of $\mathcal{T}$

have preimages

$$\begin{aligned}\rho^{-1}(1,1,1) &= \{\triangle[\pm\pi,0,0], \triangle[0,\pm\pi,0], \triangle[0,0,\pm\pi]\} = \partial(\partial(\mathcal{T}_+ \sqcup \mathcal{T}_-))\\ \rho^{-1}(e^{i2\beta},1,1) &= \{\triangle[0,\beta,\pi-\beta] \in (\partial\mathcal{T}_+)^\circ,\ \triangle[0,-\pi+\beta,-\beta] \in (\partial\mathcal{T}_-)^\circ\}\\ \rho^{-1}(1,e^{-i2\alpha},1) &= \{\triangle[\alpha,0,\pi-\alpha] \in (\partial\mathcal{T}_+)^\circ,\ \triangle[-\pi+\alpha,0,-\alpha] \in (\partial\mathcal{T}_-)^\circ\}\\ \rho^{-1}(e^{i2\beta},e^{i2\beta},1) &= \{\triangle[\pi-\beta,\beta,0] \in (\partial\mathcal{T}_+)^\circ,\ \triangle[-\beta,-\pi+\beta,0] \in (\partial\mathcal{T}_-)^\circ\}\end{aligned}$$

Thus the preimage of $(1,1,1) \in \mathcal{T}$ consists of the six vertex points of $\mathcal{T}_+ \sqcup \mathcal{T}_-$, and the preimage of every other degenerate point of $\mathcal{T}$ has two preimage points, one on $(\partial\mathcal{T}_+)^\circ$ and one on $(\partial\mathcal{T}_-)^\circ$.

Proof. By Construction (2.4.1), ρ inscribes every nondegenerate oriented triangle $\triangle[\alpha,\beta,\gamma]$ in S^1 as a triple of points $(P,Q,1)$ with the same orientation. Conversely, every such triple is obviously in the image of such a triangle. Therefore ρ is surjective and preserves orientation on nondegenerate points of $\mathcal{T}$. We will show surjectivity at the degenerate triples below by explicitly computing their preimages.

Suppose $\triangle = \triangle[\alpha,\beta,\gamma]$ is degenerate, i.e., has orientation zero. To show $\rho(\triangle)$ is degenerate is to show that two of the points $\{e^{i2\beta}, e^{-i2\alpha}, 1\}$ are the same. Since $\triangle$ is degenerate, by Definition 2.2.1 one of its interior angles α, β or γ is 0. If it's α or β we are done, because then $e^{-i2\alpha} = 1$ or $e^{i2\beta} = 1$. If it's neither then $\gamma = 0$, $\alpha + \beta = \pm\pi$, and $e^{i2\beta} = e^{-i2\alpha}$, and again we are done.

Now we check the preimages. If $\rho(\triangle[\alpha,\beta,\gamma]) = (1,1,1)$ then $2\beta = 2k\pi$ for some k, hence β is a multiple of π, and similarly for α; since the sum of all three is π, the same goes for γ. Therefore $\triangle[\alpha,\beta,\gamma]$ must be one of the six vertexes, and all, of course, map to $(1,1,1)$. The remaining degenerate triples contain exactly two distinct points: $(P,1,1), (1,Q,1)$, or $(P,P,1)$. Putting $P = e^{i2\beta}$ and $Q = e^{-i2\alpha}$ shows ρ maps the stated preimages onto the corresponding degenerate triples. Since every degenerate triangle is accounted for in some preimage, this completes the proof. □

Definition 2.4.4. Degenerate triangles in $\partial\mathcal{T}_+ \sqcup \partial\mathcal{T}_-$ are *similar* if either each has two zero interior angles, or they share one zero interior angle, and the other two angles are anti-transpositions of each other. Thus the distinct similarity classes are of form

$$\triangle[\pi,0,0] = \triangle[-\pi,0,0] = \triangle[0,\pm\pi,0] = \triangle[0,0,\pm\pi]$$

$$\triangle[0,\beta,\gamma] = \triangle[0,-\gamma,-\beta] \qquad \triangle[\alpha,0,\gamma] = \triangle[-\gamma,0,-\alpha] \qquad \triangle[\alpha,\beta,0] = \triangle[-\beta,-\alpha,0]$$

Using these identifications we immediately obtain the following.

Corollary 2.4.5. *The set of similarity classes of labeled, oriented, possibily degenerate triangles is parameterized by the abelian torus group*

$$\frac{\mathcal{T}_+ \sqcup \mathcal{T}_-}{\partial\mathcal{T}_+ \sim \partial\mathcal{T}_-} \approx \mathcal{T}$$

where the gluing of border points is determined by ρ as in Theorem 2.4.3 and Figure 2.5, identifying all six vertex points with the identity $(1,1,1) \in \mathcal{T}$, and pairs of (degenerate)

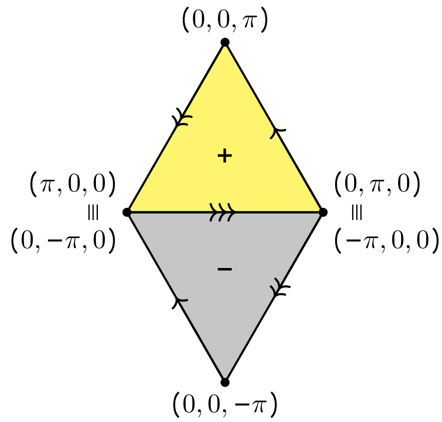


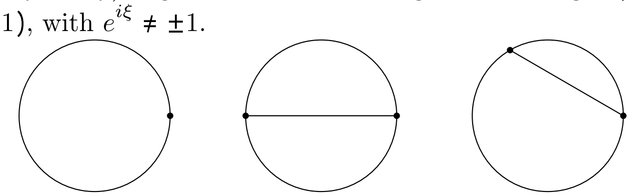
Figure 2.5. Gluing together the Triangles of Triangles

border points as

$$\begin{array}{ccccc}
\partial\mathcal{T}_+ & \longleftrightarrow & \partial\mathcal{T}_- & \stackrel{\rho}{\longmapsto} & \mathcal{T} \\
\triangle[0,\beta,\gamma] & = & \triangle[0,-\gamma,-\beta] & \longmapsto & (e^{i2\beta},1,1) \\
\triangle[\alpha,0,\gamma] & = & \triangle[-\gamma,0,-\alpha] & \longmapsto & (1,e^{-i2\alpha},1) \\
\triangle[\alpha,\beta,0] & = & \triangle[-\beta,-\alpha,0] & \longmapsto & (e^{i2\beta},e^{i2\beta},1)
\end{array}$$

2.4.6. *Image in $\mathcal{T}$ of Labeled, Oriented Triangle Types.* An element of $\mathcal{T}$ is given by a triple in S^1 whose third entry is 1. In this subsection we describe the basic types as they appear in S^1.

(a) By Definition 2.2.1 a degenerate triangle has at least one zero interior angle, so the image of its similarity class under ρ is $(e^{i\xi},1,1)$, $(1,e^{i\xi},1)$, or $(e^{i\xi},e^{i\xi},1)$, inscribed in S^1 as a chord. Specifically, on similarity classes we have:
   - The degenerate equilateral triangle has image the identity triple $(1,1,1)$.
   - The three degenerate isosceles triangles that are not equilateral have images the diameter $(-1,1,1)$, $(1,-1,1)$, or $(-1,-1,1)$.
   - The (infinitely many) degenerate scalene triangles have images $(e^{i\xi},1,1)$, $(1,e^{i\xi},1)$, or $(e^{i\xi},e^{i\xi},1)$, with $e^{i\xi}\neq\pm1$.



(b) Nondegenerate triangles have positive or negative orientation, and all angles of absolute value between 0 and π. Specifically, on $\mathcal{T}_+^\circ \sqcup \mathcal{T}_-^\circ$ we have:
   - The two nondegenerate equilateral triangles have images $(e^{i2\pi/3},e^{i4\pi/3},1)$ and $(e^{i4\pi/3},e^{i2\pi/3},1)$.
   - The (infinitely many) nondegenerate isosceles triangles have images $(e^{i\xi},e^{-i\xi},1)$, $(e^{i2\xi},e^{i\xi},1)$, or $(e^{i\xi},e^{i2\xi},1)$, depending on whether the vertex is at C, B, or A.

- The (infinitely many) nondegenerate right triangles have images $(e^{i\xi}, -1, 1)$, $(-1, e^{i\xi}, 1)$, or $(e^{i\xi}, -e^{i\xi}, 1)$, depending on whether the right angle is at A, B, or C.
- The (infinitely many) nondegenerate scalene triangles have images $(e^{i\xi_1}, e^{i\xi_2}, 1)$ not of any of the previous types.

2.5. **Distinguished Subgroups.**

2.5.1. *Subgroup Structure of Distinguished Triangle Classes.* The group structure of $\mathcal{T}$ is compatible with triangle types, in the sense that the latter form basic algebraic structures: elements of finite order, subgroups, and cosets.

Theorem 2.5.2. *The standard triangle types form the following distinguished subgroups and cosets of the torus of triangles $\mathcal{T}$.*

(a) *The three equilateral triangle classes form a group of order 3, with the two nondegenerate classes as generators and the degenerate class as the identity.*
(b) *The three degenerate nonequilateral isosceles classes each generate a group of order 2, with the degenerate equilateral class as the identity.*
(c) *The six nondegenerate right isosceles classes generate three cyclic groups of order 4, each containing a degenerate nonequilateral isosceles class (of order 2).*
(d) *The three types of isosceles triangle classes $\{\mathsf{I}_A, \mathsf{I}_B, \mathsf{I}_C\}$, distinguished by vertex, form three circle subgroups.*
(e) *The three types of degenerate classes $\{\mathsf{D}_A, \mathsf{D}_B, \mathsf{D}_C\}$, distinguished by vertex, form three circle subgroups.*
(f) *The three types of right triangle classes $\{\mathsf{R}_A, \mathsf{R}_B, \mathsf{R}_C\}$, distinguished by vertex, are cosets of the degenerate subgroups D_i, and their images are the unique elements of order two in the quotients $\mathcal{T}/\mathsf{D}_i$.*

Proof. For (d),(e), and (f) it will suffice to write down the stated subgroups; checking they form subgroups is trivial. In the following we refer to the lie algebra of relative arguments, whose fundamental domain, divided into positive and negative orientations, is Figure 2.6.

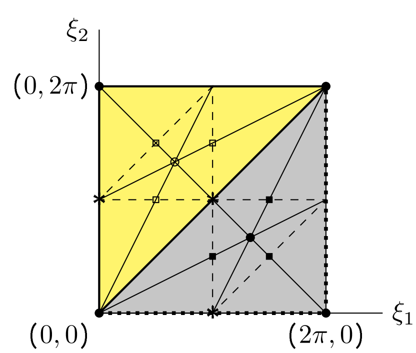


Figure 2.6. The Fundamental Domain of the Torus of Triangles

- In the plane of relative arguments, the additive groups $\mathsf{d}_A = \{(\xi, 0)\}$, $\mathsf{d}_\mathsf{B} = \{(0, \xi)\}$, and $\mathsf{d}_\mathsf{C} = \{(\xi, \xi)\}$ define three subgroups of degenerate triangles in $\mathcal{T} \equiv \sigma(\mathcal{T}) \subset \mathsf{S}^1$:

$$\mathsf{D}_A = \{(e^{i\xi}, 1, 1)\} \qquad \mathsf{D}_B = \{(1, e^{i\xi}, 1)\} \qquad \mathsf{D}_C = \{(e^{i\xi}, e^{i\xi}, 1)\}$$

- The three cosets $\mathsf{r}_A = (0, \pi) + \mathsf{d}_A$, $\mathsf{r}_B = (\pi, 0) + \mathsf{d}_B$, and $\mathsf{r}_C = (0, \pi) + \mathsf{d}_C$ correspond to the right triangles:

$$\mathsf{R}_A = (1, -1, 1) \cdot \mathsf{D}_A = \{(e^{i\xi}, -1, 1)\}$$
$$\mathsf{R}_B = (-1, 1, 1) \cdot \mathsf{D}_B = \{(-1, e^{i\xi}, 1)\}$$
$$\mathsf{R}_C = (1, -1, 1) \cdot \mathsf{D}_C = \{(e^{i\xi}, -e^{i\xi}, 1)\}$$

  The degenerate right triples comprise the set $\{(1, -1, 1), (-1, 1, 1), (-1, -1, 1)\}$.
- The subgroups $\mathsf{i}_A = \{(\xi, 2\xi)\}$, $\mathsf{i}_B = \{(2\xi, \xi)\}$, and $\mathsf{i}_C = \{(\xi, -\xi)\}$, correspond to the isosceles triangles in $\mathcal{T}$:

$$\mathsf{I}_A = \{(e^{i\xi}, e^{i2\xi}, 1)\} \qquad \mathsf{I}_B = \{(e^{i2\xi}, e^{i\xi}, 1)\} \qquad \mathsf{I}_C = \{(e^{i\xi}, e^{-i\xi}, 1)\}$$

(a): The two nondegenerate equilateral classes $\pm(e^{i2\pi/3}, e^{i4\pi/3}, 1)$ are in the intersection $\mathsf{I}_A \cap \mathsf{I}_B \cap \mathsf{I}_C$ of the isosceles subgroups, marked as the centroids of the yellow and gray triangles of Figure 2.6. They are mutually inverse, forming together with the degenerate equilateral class $(1, 1, 1)$ a subgroup of order 3.

(b): The three degenerate nonequilateral isosceles classes form the set

$$\{(e^{i\pi}, 1, 1), (1, e^{i\pi}, 1), (e^{i\pi}, e^{i\pi}, 1)\}$$

marked by $*$'s in Figure 2.6. Each is the unique order-two element of the subgroups $\mathsf{D}_A, \mathsf{D}_B, \mathsf{D}_C$, respectively.

(c): The six nondegenerate right isosceles classes form the set

$$\{\pm(e^{i\pi/2}, e^{i\pi}, 1), \pm(e^{i\pi/2}, e^{i3\pi/2}, 1), \pm(e^{i\pi}, e^{i3\pi/2}, 1)\}$$

marked with square points in Figure 2.6. Each has order 4 and square equal to one of the three degenerate nonequilateral isosceles classes. □

Remark 2.5.3. The symmetry of the plane of relative arguments suggests some other families not included in the standard taxonomies:

- The perpendicular "anti-isosceles" subgroups $\mathsf{i}_A^\perp = \{(-2\xi, \xi)\}$, $\mathsf{i}_B^\perp = \{(\xi, -2\xi)\}$, and $\mathsf{i}_C^\perp = \mathsf{d}_C = \{(\xi, \xi)\}$ define subgroups

$$\mathsf{I}_A^\perp = \{(e^{-i2\xi}, e^{i\xi}, 1)\} \qquad \mathsf{I}_B^\perp = \{(e^{i\xi}, e^{-i2\xi}, 1)\} \qquad \mathsf{I}_C^\perp = \mathsf{D}_C = \{(e^{i\xi}, e^{i\xi}, 1)\}$$

- The distinguished coset $(0, \pi) + \mathsf{i}_C = \{(\xi, \pi - \xi)\}$ defines the coset of "anti-right" triangles.

$$(1, -1, 1) \cdot \mathsf{I}_C = \{(e^{i\xi}, -e^{-i\xi}, 1)\}$$

When graphed in Figure 2.6, the above objects make a complete graph with nine vertices.

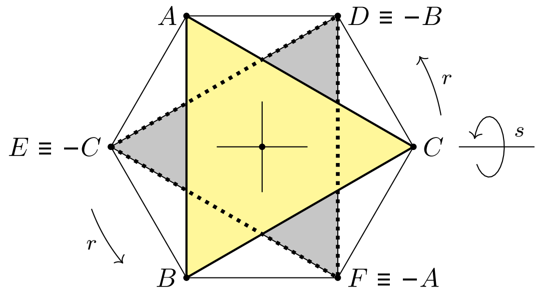
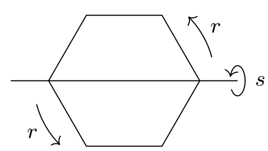


Figure 2.7. D_6 Acting on Similarity Classes

2.6. **Absolute Similarity Classes.** Let

$$\pm S_3 := S_3 \times \langle \pm e \rangle = \{\pm e, \pm(123), \pm(132), \pm(23), \pm(13), \pm(12)\}$$

a group of order 12 isomorphic to the dihedral group $D_6 = \langle r, s \rangle$, where in our notation $r \equiv -(123)$ is counterclockwise rotation of a regular hexagon by $\pi/3$, $r^3 \equiv -e$, and $s \equiv -(12)$ is reflection about the horizontal axis:



The permutation representation

$$\pm S_3 \longrightarrow \mathrm{GL}_3(\mathbb{R})$$

defines an action on $\mathbb{R}^3$ that stabilizes Δ. Explicitly,

$$\pm S_3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(\pm\sigma, (\theta_1, \theta_2, \theta_3)) \longmapsto (\pm\theta_{\sigma(1)}, \pm\theta_{\sigma(2)}, \pm\theta_{\sigma(3)})$$

Since both δ and the splitting map σ are $\mathbb{R}$-linear, the induced representation $\pm S_3 \longrightarrow \mathrm{GL}_2$ acts on the plane of relative arguments $\mathbb{R}^2$. Explicitly, $\pm\sigma(\theta_1 - \theta_3, \theta_2 - \theta_3) = \pm(\theta_{\sigma(1)} - \theta_{\sigma(3)}, \theta_{\sigma(2)} - \theta_{\sigma(3)})$. The image of the corresponding two-dimensional permutation representation is

$$\pm e = \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \pm(123) = \pm\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \qquad \pm(132) = \pm\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$
$$\pm(12) = \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \pm(13) = \pm\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \pm(23) = \pm\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

Since these actions do not change the (unordered) absolute values of the arguments in a triple, they stabilize the kernel of exp, and induce actions on $(\mathsf{S}^1)^3$ and $\mathcal{T}$. On S^1 negation acts as complex conjugation, which reverses the orientation of a nondegenerate triple since it turns an ordering that goes counterclockwise to one that goes clockwise.

The induced action on interior angles in $\frac{\mathcal{T}_+ \sqcup \mathcal{T}_-}{\sim}$ is represented by Figure 2.7, in which the triangles of triangles are inscribed in a regular hexagon. The A, B, C-axes and $-A, -B, -C$-axes are the same as those in Figure 2.1, and the gluing is $AB \leftrightarrow DF, BC \leftrightarrow$

$ED, AC \leftrightarrow EF$. The 2×2 matrix action on relative arguments shows the induced action is given by the map

$$\begin{array}{ccccc}
\pm S_3 & \longleftrightarrow & D_6 & \longmapsto & S_6 \\
-(123) & \longleftrightarrow & r & \longmapsto & (AEBFCD) \\
-(12) & \longleftrightarrow & s & \longmapsto & (AB)(DF) \\
-e & \longleftrightarrow & r^3 & \longmapsto & (AF)(BD)(CE) \\
(123) & \longleftrightarrow & r^4 & \longmapsto & (ACB)(DFE) \\
(12) & \longleftrightarrow & r^3 s & \longmapsto & (AD)(BF)(CE) \\
(13) & \longleftrightarrow & r^5 s & \longmapsto & (AE)(BD)(CF) \\
(23) & \longleftrightarrow & rs & \longmapsto & (AF)(BE)(CD) \\
-(13) & \longleftrightarrow & r^2 s & \longmapsto & (AC)(EF) \\
-(23) & \longleftrightarrow & r^4 s & \longmapsto & (BC)(DE)
\end{array}$$

The elements that preserve orientation form the subgroup

$$D_3 \simeq \langle r^2, s \rangle \leq \pm S_3$$

which then acts separately on $\mathcal{T}_+$ and $\mathcal{T}_-$.

Theorem 2.6.1. *The orbit space* $[\mathcal{T}]$ *of* $\mathcal{T}$ *under* $\pm S_3 \simeq D_6$ *is the set of similarity classes of triangles.*

Proof. Let $\triangle_1 = \triangle[\theta_1, \theta_2, \theta_3]$ and $\triangle_2 = \triangle[\psi_1, \psi_2, \psi_3]$. Write $\triangle_1 \simeq \triangle_2$ for similar, and $[\triangle_1] = [\triangle_2]$ for same orbit. Then

$$\begin{aligned}
\triangle_1 \simeq \triangle_2 &\iff \pm\{\theta_1, \theta_2, \theta_3\} = \{\psi_1, \psi_2, \psi_3\} \iff \exists \sigma : (\psi_1, \psi_2, \psi_3) = \pm(\theta_{\sigma(1)}, \theta_{\sigma(2)}, \theta_{\sigma(3)}) \\
&\iff [\triangle_1] = [\triangle_2]
\end{aligned}$$

□

2.6.2. *Triangle Multiplicities.* The differences in symmetry distinguish the scalene, right, isosceles, and equilateral degenerate and nondegenerate triangles, and this is measured by the sizes of their stabilizer subgroups.

Definition 2.6.3. The *multiplicity* $\mathrm{mult}(\triangle)$ of $\triangle \in \mathcal{T}$ is the order of $\mathrm{stab}(\triangle) \leq \pm S_3 \simeq D_6$.

Theorem 2.6.4. *The multiplicities of similarity classes of labeled, oriented, possibly degenerate triangles are as follows.*

(a) *The degenerate equilateral triangle has multiplicity* 12.
(b) *The two nondegenerate equilateral triangles have multiplicity* 6.
(c) *The three degenerate nonequilateral isosceles triangles have multiplicity* 4.
(d) *Degenerate scalene triangles each have multiplicity* 2.
(e) *Nondegenerate nonequilateral isosceles triangles each have multiplicity* 2.
(f) *Nondegenerate scalene triangles each have multiplicity* 1.

Proof. We refer to Figure 2.7 and compute using D_6.

The degenerate equilateral $\triangle_e$ is represented by any of the vertices in Figure 2.7, hence its orbit has lenght 1, and it is fixed by D_6. Therefore $\mathrm{mult}(\triangle_e) = 12$.

The nondegenerate equilateral triangles $\triangle_{E_+}$ and $\triangle_{E_-}$ are conjugate by the element $r^3 s$, and each has stabilizer $\langle r^2, s\rangle \simeq D_3$, hence $\mathrm{mult}(\triangle_{E_+}) = \mathrm{mult}(\triangle_{E_-}) = 6$.

The degenerate nonequilateral isosceles triangles $\left\{\triangle[0, \frac{\pi}{2}, \frac{\pi}{2}], \triangle[\frac{\pi}{2}, 0, \frac{\pi}{2}], \triangle[\frac{\pi}{2}, \frac{\pi}{2}, 0]\right\}$ are conjugate by rotations. We compute $\mathrm{stab}(\triangle[\frac{\pi}{2}, \frac{\pi}{2}, 0]) = \left\langle r^3, s\right\rangle \simeq C_2 \times C_2$, hence $\mathrm{mult}(\triangle[\frac{\pi}{2}, \frac{\pi}{2}, 0]) = 4$, and the others follow suit.

Degenerate scalene triangles have orbits of length 6 under the elements of $\langle r\rangle$. The stabilizer of $\triangle = \triangle[\alpha, \beta, 0]$ (with $\alpha, \beta \neq 0, \pm\frac{\pi}{2}, \pm\pi$) is $\mathrm{stab}(\triangle) = \langle r^3 s\rangle$, hence $\mathrm{mult}(\triangle) = 2$.

Nondegenerate nonequilateral isosceles triangles have orbits of length 6 under $\langle r\rangle$. If $\triangle \in \mathsf{I}_C$ then $\mathrm{stab}(\triangle) = \langle s\rangle$, hence $\mathrm{mult}(\triangle) = 2$.

Nondegenerate scalene triangles $\triangle$ have orbits of length 12, and stabilizers $\mathrm{stab}\left(\triangle(\alpha, \beta, \gamma)\right) = \langle e\rangle$, where α, β, γ are distinct. Therefore $\mathrm{mult}(\triangle) = 1$. □

2.7. **Measure.** By Theorem 2.6.1 the similarity classes of triangles are repeated in $\mathcal{T}$ according to their multiplicity under the action of D_6. This motivates the following definition.

Definition 2.7.1. Let $\mathsf{F} \subset \mathcal{T}$ be a family of similarity classes of labeled, oriented triangles, and let $[\mathsf{F}]$ denote the corresponding set of similarity classes in the orbit space $[\mathcal{T}] \equiv \mathcal{T}/D_6$. The *relative measure* of $[\mathsf{F}]$ is

$$\mu([\mathsf{F}]) = \mathrm{mult}(\triangle) \cdot |\mathsf{F}|$$

where $|\mathsf{F}|$ is the Euclidean measure in $\mathcal{T}$ under the metric of $\mathcal{T}$ defined by $\rho : \mathcal{T}_+ \sqcup \mathcal{T}_- \longrightarrow \mathcal{T}$ in Definition 2.4.2, and $\triangle \in \mathsf{F}$ is a generic element.

The relative measure of a two-parameter family is given by its Euclidean measure, since all generic points of such families have multiplicity one.

Theorem 2.7.2. *Let* $\mathsf{O}, \mathsf{A}, \mathsf{I}, \mathsf{R}, \mathsf{D}, \mathsf{OI}, \mathsf{AI}$ *be the families of similarity classes of labeled, oriented triangles in* $\mathcal{T}$ *that are obtuse, acute, isosceles, right, degenerate, obtuse isosceles, and acute isosceles, respectively. Then the relative measures are*

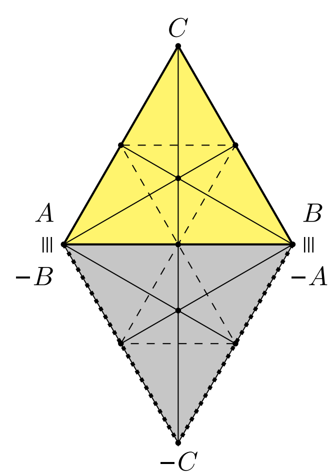


Figure 2.8. The Flat Torus $\mathcal{T}$

(a) $\mu(\mathcal{T}) = \sqrt{3}\pi^2$; $\mu(\mathsf{O}) = \frac{3\sqrt{3}}{4}\pi^2$; $\mu(\mathsf{A}) = \frac{\sqrt{3}}{4}\pi^2$;
(b) $\mu(\mathsf{I}) = 6\sqrt{6}\pi$; $\mu(\mathsf{R}) = 3\sqrt{2}\pi$; $\mu(\mathsf{D}) = 6\sqrt{2}\pi$; *and* $\mu(\mathsf{OI}) = \mu(\mathsf{AI}) = 3\sqrt{6}\pi$.

In particular we have the following ratios:

$$[\mathsf{O} : \mathsf{A}] = [3 : 1] \qquad [\mathsf{I} : \mathsf{AI}] = [\mathsf{I} : \mathsf{OI}] = [2 : 1] \qquad [\mathsf{I} : \mathsf{R}] = [2\sqrt{3} : 1] \qquad [\mathsf{D} : \mathsf{R}] = [2 : 1]$$

Proof. Using Figure 2.8 we easily compute $|\mathsf{T}| = \sqrt{3}\pi^2$, $|\mathsf{O}| = \frac{3\sqrt{3}}{4}\pi^2$, and $|\mathsf{A}| = \frac{\sqrt{3}}{4}\pi^2$. Generic elements of these two-parameter families all have multiplicity one, so these are the relative measures as defined in Definition 2.7.1.

For the one-parameter families we compute $|\mathsf{I}| = 3\sqrt{6}\pi$, $|\mathsf{R}| = 3\sqrt{2}\pi$, $|\mathsf{D}| = 3\sqrt{2}\pi$, $|\mathsf{OI}| = \frac{3\sqrt{6}}{2}\pi$, and $|\mathsf{AI}| = \frac{3\sqrt{6}}{2}\pi$. Generic degenerate elements, which are scalene, and generic nondegenerate isosceles both have multiplicity two, and a generic right triangle has multiplicity one, by Theorem 2.6.4. Therefore the relative measures are as stated in these cases. For obtuse and acute isosceles triangles the multiplicites are 2, so they have the same relative measure $3\sqrt{6}\pi$. The ratios follow immediately. ☐

3. The Sphere of Triangles

One natural way to consider triangles is as triples of vertices. We show that the parameter space of triangles up to similarity in this notion is a sphere. This is known result (see [ES15, CNSS19]) that we discovered independently, and we provide the explicit description for the sake of concreteness.[2]

3.1. **Terminology & Definitions.**

Definition 3.1.1. A *triangle* is a point $(A, B, C) \in \mathbb{C}^3$, where each coordinate is a vertex of the triangle in $\mathbb{C}$.

Thus $\mathbb{C}^3$ is a parameter space of labeled triangles. But $\mathbb{C}^3$ includes points not traditionally thought of as plane triangles.

Definition 3.1.2. A triangle is *degenerate* if its area is zero. A triangle is *nondegenerate* if it has nonzero area.

Moreover, a degenerate triangle $(A, B, C) \in \mathbb{C}^3$ is of

(a) *multiplicity 1* if A, B, C are distinct collinear points.
(b) *multiplicity 2* if exactly two of the vertices are equal.
(c) *multiplicity 3* if all three vertices are equal.

A triangle is *nontrivial* if it is not multiplicity 3 degenerate. See Figure 1.1.

Now we understand $\mathbb{C}^3$ as a space of degenerate and nondegenerate triangles. To formalize our notion of similarity, we must first make another definition.

Definition 3.1.3. A nondegenerate triangle $(A, B, C) \in \mathbb{C}^3$ is *positively (negatively) oriented* if the curve $A \to B \to C \to A$ of line segments in the complex plane is positively (negatively) oriented.

Now we are ready to consider *similarity classes* of triangles in $\mathbb{C}^3$.

Definition 3.1.4. Let $p = (A, B, C) \in \mathbb{C}^3$ be a triangle. Let $V_0 = \mathbb{C} \cdot (1, 1, 1)$, and let $V_p = \mathbb{C} \cdot p$. The *(labeled) similarity class* of p is

$$[p] = (V_0 + V_p) - V_0.$$

Equivalently, $[p] = \{zp + t(1, 1, 1) : z \in \mathbb{C}^\times, t \in \mathbb{C}\}$.

V_0 corresponds to all possible translations of p, and V_p corresponds to all possible rotations and dilations of p, which is captured by complex scaling. Since this construction does not allow for reflection, we are considering *direct* similarity. We remove V_0 as nondegenerate triangles should not be similar to degenerate triangles.

---

[2]This section is the result of the a 2023 Frost Summer Undergraduate Research Program project supervised by Eric Brussel that myself, Elijah Guptill, and Kelly Lyle participated in.

3.2. **Group action on $[p]$.** We imagine two triangles to be similar to each other if one can be scaled, rotated, and translated to agree with the other. We formalize this notion using group actions.

Consider the group $\mathbb{C} \rtimes \mathbb{C}^{\times}$, with group action

$$(\tau, z)(\tau', z') = (\tau\phi_z(\tau'), zz'),$$

where $h : \mathbb{C}^* \to \mathrm{Aut}(\mathbb{C})$ is given by $h(z) = \phi_z$, and $\phi_z(\tau) = z\tau$.

$\mathbb{C} \rtimes \mathbb{C}^*$ acts on similarity classes as follows:

$$\mathbb{C} \rtimes \mathbb{C}^* \times [p] \to [p]$$
$$((\tau, z), (A, B, C)) \mapsto (zA + \tau, zB + \tau, zC + \tau)$$

Proposition 3.2.1. *Let $p \in \mathbb{C}^3$ be a nontrivial triangle. Then $[p]$ is a principal homogeneous space for $\mathsf{Sim}(\mathbb{R}^2) \simeq \mathbb{C} \rtimes \mathbb{C}^*$.*

Proof. Suppose $(\tau, z) \cdot (A, B, C) = (A, B, C)$ for some $(A, B, C) \in [p]$. By the group action defined above, we see that $(zA + \tau, zB + \tau, zC + \tau) = (A, B, C)$, and so

$$\tau = A(1 - z) = B(1 - z) = C(1 - z).$$

If $z = 1$, then $\tau = 0$. If $z \neq 1$, it follows that $A = B = C$, which contradicts the assumption that $[p]$ is a similarity class of nontrivial triangles. Thus $(\tau, z) = (0, 1)$, and so the action is free. From the definition of $[p]$, it is clear that the action is transitive. □

3.3. **Space of Triangles up to Similarity.** We have established that a given similarity class of a nontrivial triangle p is a two $\mathbb{C}$-dimsensional subset of $\mathbb{C}^3$. To construct our space of similarity classes, we first mod out by translation, as follows. This is a short exact sequence.

$$0 \longrightarrow V_0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}/V_0 \longrightarrow 0$$

We apply the dual functor, which gives us the following sequence of dual spaces.

$$0 \longleftarrow (V_0)^* \longleftarrow \mathbb{C}^* \longleftarrow (\mathbb{C}/V_0)^* \longleftarrow 0$$

The elements of $(\mathbb{C}^3/V_0)^*$ are maps on $\mathbb{C}^3$ that are well-defined on V_0, and so the kernel of these maps must be V_0. $(\mathbb{C}^3/V_0)^*$ is precisely the plane $x + y + z = 0$ in $\mathbb{C}^3$, the plane with normal vector $(1, 1, 1)$.

To summarize, modding out by translation amounts to the following map

$$\mathbb{C}^3 \to (\mathbb{C}^3/V_0)^*$$
$$(A, B, C) \mapsto (B - C, C - A, A - B)$$

where the image of (A, B, C) in $(\mathbb{C}^3/V_0)^*$ is $(A, B, C) \times (1, 1, 1)$. This is a linear transformation given by multiplication on the left by

$$D = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Since $\dim_{\mathbb{C}}(\mathbb{C}^3/V_0)^* = 2$, we would like to map to $\mathbb{C}^2$. We do so by multiplying on the left by $M \in SO(3)$ given by

$$M = \begin{bmatrix} \frac{1}{6}(3+\sqrt{3}) & \frac{1}{6}(-3+\sqrt{3}) & -\frac{1}{\sqrt{3}} \\ \frac{1}{6}(-3+\sqrt{3}) & \frac{1}{6}(3+\sqrt{3}) & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{3.3.0.1}$$

which rotates $\frac{1}{\sqrt{3}}(1,1,1)$ to $(0,0,1)$, and then only considering the first two coordinates. Since this is a linear isometry, it will not "disturb" the similarity classes, so to speak.

We have constructed $\mathbb{C}^2$ as the set of labeled triangles up to translation. To construct our space of similarity classes, we must mod out by rotation and dilation, which amounts to modding out by complex scaling. Fortunately, this is quite simple, as we simply consider the set of all complex lines through the origin in $\mathbb{C}^2$, known as $\mathrm{Gr}(1,\mathbb{C}^2)$.

It is known that $\mathrm{Gr}(1,\mathbb{C}^2) = P^1(\mathbb{C}) \simeq S^2$, so our moduli space of labeled triangles up to similarity is a 2-sphere. To construct an explicit map, we endow $\mathbb{C}^2$ with the algebraic structure of the quaternions $\mathbb{H}$.

The quaternions are a 4-dimensional real vector space with basis $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$, and they can be multiplied in the sense that

$$\mathbf{ij} = \mathbf{k}, \qquad \mathbf{jk} = \mathbf{i}, \qquad \mathbf{ki} = \mathbf{j}.$$

Thus they form a 4-dimensional $\mathbb{R}$-algebra, with the real quaternions coinciding with the center.

We can construct $\mathbb{H}$ from $\mathbb{C}^2$ using the Cayley-Dickinson construction [Bae02, Sec. 2.2]. This will give us a canonical identification between $\mathbb{C}^2$ and $\mathbb{H}$ Explicitly, define the map

$$\mathbb{C}^2 \to \mathbb{H}$$
$$(a,b) \mapsto a + b\mathbf{j}.$$

Addition is component-wise, and multiplication is defined by

$$(a,b)(c,d) = (ac - b\overline{d}, ad + b\overline{c})$$

The norm of (a,b) is $|a|^2 + |b|^2$, and we see that

$$(a,b)^{-1} = \frac{(\overline{a}, -b)}{|(a,b)|^2}.$$

One can check that these operations endow $\mathbb{C}^2$ with the algebraic structure of $\mathbb{H}$. Now for our explicit map.

Definition 3.3.1. The *Hopf map* $h : \mathbb{C}^2 \to \mathbb{C}^2$ is a left-adjoint action on $\mathbb{C}^2$. On quaternions, this is the map $r \mapsto r^{-1}\mathbf{i}r$. Explicitly, h is given by

$$h : \mathbb{C}^2 \to \mathbb{C}^2$$
$$(a,b) \mapsto (a,b)^{-1}(\mathbf{i},0)(a,b).$$

Note that this is precisely the famous Hopf fibration [Lyo03].

Lemma 3.3.2. *The Hopf map h is well-defined on* $\mathrm{Gr}(1,\mathbb{C}^2)$.

Proof. Let $(a,b) \in \mathbb{C}^2$, and let $\lambda \in \mathbb{C}$. The scalar λ embed in $\mathbb{C}^2$ as $(\lambda, 0)$. Then

$$\begin{aligned} h((\lambda,0)(a,b)) &= [(\lambda,0)(a,b)]^{-1}(\mathbf{i},0)[(\lambda,0)(a,b)] \\ &= [(\lambda a, \lambda b)]^{-1}(\mathbf{i},0)(\lambda a, \lambda b) \\ &= \frac{(\overline{\lambda a}, -\lambda b)(\mathbf{i},0)(\lambda a,\lambda b)}{|(\lambda a,\lambda b)|^2} \\ &= \frac{|\lambda|^2((|a|^2-|b|^2)\mathbf{i}, 2\overline{a}b\mathbf{i})}{|\lambda|^2|(a,b)|^2} \\ &= \frac{((|a|^2-|b|^2)\mathbf{i}, 2\overline{a}b\mathbf{i})}{|(a,b)|^2} \\ &= (a,b)^{-1}(\mathbf{i},0)(a,b) \\ &= h((a,b)). \end{aligned}$$

□

Observe that the image of $\mathbb{C}^2$ under h is contained in the pure imaginary quaternions $\mathrm{Im}(\mathbb{H})$, since $\mathrm{Re}((|a|^2 - |b|^2)\mathbf{i}) = 0$.

Since the norm defined above is multiplicative,

$$|h(a,b)| = |(a,b)^{-1}(i,0)(a,b)| = |(\mathbf{i},0)| = 1.$$

Therefore the image of $\mathbb{C}^2$ under h is the imaginary quaternions of unit length. This set is a sphere S^2. To realize this S^2 as a subset of Euclidean space, we use the canonical map

$$\begin{aligned} \mathrm{Im}(\mathbb{H}) &\to \mathbb{R}^3 \\ (\mathbf{i},\mathbf{j},\mathbf{k}) &\mapsto (\mathbf{x},\mathbf{y},\mathbf{z}). \end{aligned}$$

Summarizing, we see that we have proved the following theorem.

Theorem 3.3.3. *The set of similarity classes of triangles when considered as ordered triples of vertices is a sphere. An explicit parameterization is given by*

$$\begin{aligned} p : \mathbb{C}^3 &\to \mathbb{R}^3 \\ (A,B,C) &\mapsto i(h(M(B-C, C-A, A-B))), \end{aligned}$$

where M is the linear isometry in Equation 3.3.0.1, h is the Hopf map from Definition 3.3.1, and $i : \mathrm{Im}(\mathbb{H}) \to \mathbb{R}^3$ is the canoncial isomorphism between the pure imaginary quaternions and $\mathbb{R}^3$. Moreover, $p(\mathbb{C}^3) = S^2$ in $\mathbb{R}^3$.

Remark 3.3.4. We compute the stabilizer subgroups of our similarity classes $r \in S^2$. Since the Hopf Map is a conjugation map, these are precisely the centralizer subgroups $C(r)$, the set of elements in $\mathbb{H}$ that commute with r.

Proposition 3.3.5. *Let $r \in S^2 \subset \mathrm{Im}(\mathbb{H})$. The centralizer $C(r)$ of r is a field isomorphic to $\mathbb{C}$.*

Proof. One can represent quaternions in scalar-vector form, where $(t,v) \in \mathbb{R}\times\mathbb{R}^3$. Multiplication of quaternions is then given by

$$(t_1,v_1)(t_2,v_2) = (t_1t_2 - v_1\cdot v_2, t_1v_2 + t_2v_1 + v_1 \times v_2),$$

where we use the standard dot and cross products in $\mathbb{R}^3$.

Let $(t, v) \in \mathbb{R} \times \mathbb{R}^3 \simeq \mathbb{H}$. Write $r = (0, u)$, where $|u| = 1$. Then

$$\begin{aligned} (t,v) \in C(r) &\iff (0,u)(t,v) = (t,v)(0,u) \\ &\iff (-u \cdot v, tu + u \times v) = (-u \cdot v, tu + v \times u) \\ &\iff u \times v = v \times u \\ &\iff u \text{ and } v \text{ are parallel.} \end{aligned}$$

It follows that $v = \lambda u$, where $\lambda \in \mathbb{R}$. Consider the function

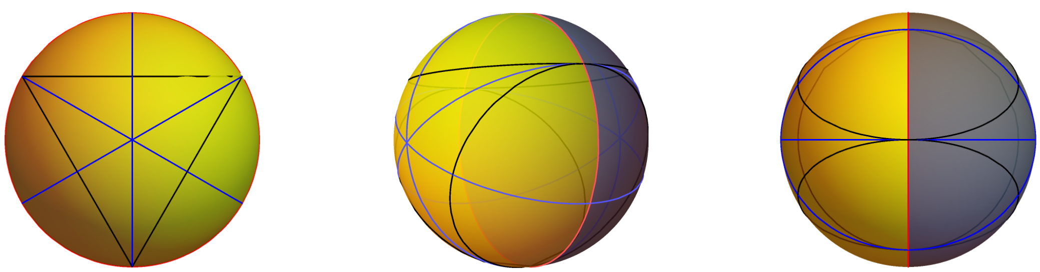
$$\begin{aligned} f : C(r) &\to \mathbb{C} \\ (t, \lambda u) &\mapsto t + \lambda \mathbf{i} \end{aligned}$$

We see that f is a group homomorphism. To see that f is a ring homomorphism, let $(s, \lambda u), (t, \mu u) \in C(r)$. Since u is a unit vector, it follows that

$$\begin{aligned} f((s,\lambda u)(t,\mu u)) &= f(st - (\lambda\mu)u \cdot u, s\mu u + t\lambda u + (\lambda\mu)u \times u) \\ &= f(st - \lambda\mu, (s\mu + t\lambda)u) \\ &= (st - \lambda\mu) + (s\mu + t\lambda)\mathbf{i} \\ &= (s + \lambda\mathbf{i})(t + \mu\mathbf{i}) \\ &= f(s, \lambda u) f(t, \mu u). \end{aligned}$$

f is clearly bijective, and so $C(r)$ is isomorphic to the field $\mathbb{C}$. □

3.4. **Distinguished Families on the Sphere.** We visualize the sphere of triangles using Mathematica in Figure 3.1 and make some observations.



(a) xz-plane (b) Side view (c) xy-plane

Figure 3.1. The Sphere of Triangles. The positively and negatively oriented triangles lie on the yellow and gray hemispheres, respectively. The isosceles, right, and degenerate triangles lie on the blue, black, and red curves, respectively.

Proposition 3.4.1. *Relative to Figure 3.1, we see that*

(a) The degenerate triangles (red curve) lie on a great circle. The positively and negatively orientated triangles lie on the yellow and gray hemispheres, respectively.
(b) The three blue curves of isosceles triangles correspond to which of the three vertices is the isosceles vertex of the triangle. They are each great circles passing through both poles.

(c) The positively oriented equilateral triangle sits at the point $(-1,0,0)$ *the intersection of the three blue isosceles curves. The negatively oriented equilateral triangle is at* $(1,0,0)$.
(d) The right triangles (black curves) trace out circles of radius $\frac{\sqrt{3}}{2}$ *that bound spherical caps.*
(e) The obtuse triangles lie on the spherical caps bound by the right triangles. The acute triangles lie in the spherical triangles at each pole.
(f) The six right isosceles triangles are at the six intersection points of the isosceles (blue) and right (black) curves.
(g) The three multiplicity 2 degenerate triangles lie at the three tangent points of the black circles, which lie on the degenerate red curve as well.
(h) The three multiplicity 1 degenerate triangles where the center vertex is centered between the two other vertices lie at the three points where only the blue and red curves intersect.

Proof. Let D; $\mathsf{R_A}, \mathsf{R_B}, \mathsf{R_C}$; $\mathsf{I_A}, \mathsf{I_B}, \mathsf{I_C}$ be the labeled, oriented similarity classes of degenerate; right triangles with right angle at vertex A, B, C; and isosceles triangles with the vertex angle at vertex A, B, C, respectively.

We get the following parameterizations of certain distinguished families of triangles. For the parameterization, let $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, where we define the value at infinity to be the limit as $t \to \infty$. We compute that

$$\begin{aligned}
\mathsf{D} &= \{[(0,t,1)] : t \in \hat{\mathbb{R}}\},\\
\mathsf{I_A} &= \{[(t,i,-i)] : t \in \hat{\mathbb{R}}\},\\
\mathsf{I_B} &= \{[(-i,t,i)] : t \in \hat{\mathbb{R}}\},\\
\mathsf{I_C} &= \{[(i,-i,t)] : t \in \hat{\mathbb{R}}\},\\
\mathsf{R_A} &= \{[(0,t,i)] : t \in \hat{\mathbb{R}}\},\\
\mathsf{R_B} &= \{[(i,0,t)] : t \in \hat{\mathbb{R}}\}, \text{ and}\\
\mathsf{R_C} &= \{[(t,i,0)] : t \in \hat{\mathbb{R}}\}.
\end{aligned}$$

Using the map p from Theorem 3.3.3, we compute the images of these families on S^2 in $\mathbb{R}^3$, and find that

$$p(\mathsf{D}) = \left\{ \left(\frac{\sqrt{3}(t^2-2t)}{2(t^2-t+1)}, 0, \frac{t^2+2t-2)}{2(t^2-t+1)} \right) \in \mathbb{R}^3 : t \in \hat{\mathbb{R}} \right\},$$

$$p(\mathsf{I_A}) = \left\{ \left(-\frac{\sqrt{3}(t^2-3)}{2(t^2+3)}, -\frac{2\sqrt{3}t}{t^2+3}, \frac{t^2-3}{2(t^2+3)} \right) \in \mathbb{R}^3 : t \in \hat{\mathbb{R}} \right\},$$

$$p(\mathsf{I_B}) = \left\{ \left(\frac{\sqrt{3}(t^2-3)}{2(t^2+3)}, -\frac{2\sqrt{3}t}{t^2+3}, \frac{t^2-3}{2(t^2+3)} \right) \in \mathbb{R}^3 : t \in \hat{\mathbb{R}} \right\},$$

$$p(\mathsf{I_C}) = \left\{ \left(0, -\frac{2\sqrt{3}t}{t^2+3}, \frac{t^2-3}{t^2+3} \right) \in \mathbb{R}^3 : t \in \hat{\mathbb{R}} \right\},$$

$$p(\mathsf{R_A}) = \left\{ \left(\frac{\sqrt{3}t^2}{2(t^2+1)}, -\frac{\sqrt{3}t}{t^2+1}, \frac{t^2-2}{2(t^2+1)} \right) \in \mathbb{R}^3 : t \in \hat{\mathbb{R}} \right\},$$

$$p(\mathsf{R_B}) = \left\{ \left(-\frac{\sqrt{3}}{2(t^2+1)}, -\frac{\sqrt{3}t}{t^2+1}, \frac{-2t^2+1}{2(t^2+1)} \right) \in \mathbb{R}^3 : t \in \hat{\mathbb{R}} \right\}, \text{ and}$$

$$p(\mathsf{R_C}) = \left\{ \left(-\frac{\sqrt{3}(t^2-1)}{2(t^2+1)}, -\frac{\sqrt{3}t}{t^2+1}, \frac{1}{2} \right) \in \mathbb{R}^3 : t \in \hat{\mathbb{R}} \right\}.$$

It is easy to see that $p(\mathsf{D})$ lies on the plane $y = 0$. Since this is a plane through the origin in $\mathbb{R}^3$, we conclude that $p(\mathsf{D})$ is a *great circle* of the sphere, which proves observation a.

Similarly, observe that $p(\mathsf{I_A}), p(\mathsf{I_B})$, and $p(\mathsf{I_C})$ lie on the planes $x = -\sqrt{3}z$, $x = \sqrt{3}z$, and $x = 0$, respectively. Since these are all planes through the origin, we conclude that the isosceles families are great circles of the sphere, proving observation b. To see the equilateral points, we compute that $p(1, \omega, \omega^2) = (0, 0, -1)$ and $p(1, \omega^2, \omega) = (0, 0, 1)$, where $\omega = e^{2\pi i/3}$. Note that both points lie on $p(\mathsf{I}_A), p(\mathsf{I}_B)$, and $p(\mathsf{I}_A)$, proving observation c.

Observe that $p(\mathsf{R_A}), p(\mathsf{R_B}), p(\mathsf{R_C})$ lie on the planes $z = \sqrt{3}x - 1$, $z = -\sqrt{3}x - 1$, and $z = \frac{1}{2}$, respectively. Furthermore, $p(\mathsf{R_A})$ and $p(\mathsf{R_B})$ both satisfy the equation $y^2 + \frac{4}{3}(z + \frac{1}{4})^2 = \frac{3}{4}$, which means they trace out circles of radius $\frac{\sqrt{3}}{2}$ centered at $(\pm\frac{\sqrt{3}}{4}, 0, -\frac{1}{4})$, respectively. $p(\mathsf{R_C})$ is much easier to recognize; it satisfies the equation $x^2 + y^2 = \frac{3}{4}$, thus tracing out a circle of radius $\frac{\sqrt{3}}{2}$ from center $(0, 0, \frac{1}{2})$. This proves observation d.

Note that the equilateral triangles lie outside the three spherical caps bounded by $p(\mathsf{R_A})$, $p(\mathsf{R_B})$, and $p(\mathsf{R_C})$. By the logic of the projection, observation e holds.

Observations f, g, and h follow by simple computations of where p sends representative triangles in the respective similarity classes.

□

3.5. **The Measure.** The sphere S^2 has a standard spherical area measure, which induces a measure on the set of similarity classes.

Definition 3.5.1. Let F be a family of similarity classes of triangles. Then the measure $\mu(\mathsf{F})$ of F is the standard spherical measure on S^2. If $\dim \mathsf{F} = 1$, this is great circle distance. If $\dim \mathsf{F} = 2$, this is surface area on the sphere.

Since our space of labeled similarity classes of triangles, S^2, is compact, we can put the uniform measure on it, which allows us to answer the original question posed by Lewis Carroll: *What is the probability that a random triangle is obtuse?* [Woo61].

Theorem 3.5.2. *Let* $\mathsf{T}, \mathsf{A}, \mathsf{O}, \mathsf{D}, \mathsf{I}, \mathsf{R}, \mathsf{AI}, \mathsf{OI}$ *be the families all, acute, obtuse, degenerate, isosceles, right, acute isosceles, and obtuse isosceles triangles, respectively. Then*

(a) $\mu(\mathsf{T}) = 4\pi$, $\mu(\mathsf{A}) = \pi$, $\mu(\mathsf{O}) = 3\pi$, *and*
(b) $\mu(\mathsf{D}) = 2\pi$, $\mu(\mathsf{I}) = 6\pi$, $\mu(\mathsf{R}) = 3\sqrt{3}\pi$, $\mu(\mathsf{AI}) = 4\pi$, $\mu(\mathsf{OI}) = 2\pi$.

It follows that

$$[\mathsf{O} : \mathsf{A}] = 3 : 1 \quad [\mathsf{OI} : \mathsf{AI}] = 1 : 2 \quad [\mathsf{I} : \mathsf{R}] = 2 : \sqrt{3} \quad [\mathsf{D} : \mathsf{R}] = 2 : 3\sqrt{3}.$$

Proof. The surface area of S^2 is 4π. Since the obtuse triangles are mapped to the three spherical caps bounded by the right triangles, it suffices to compute the area of said caps. The height of each cap is $\frac{1}{2}$, and so the surface area of an individual cap is π. There are three such obtuse caps, and their total area is 3π. The remaining area is acute triangles.

Observe that D is a single great circle, and I is three distinct great circles. R is three distinct circles of radius $\frac{\sqrt{3}}{2}$. $\mu(\mathsf{OI})$ is the length of the great circles passing through the spherical caps of height $\frac{1}{2}$.

The ratios are computed as is appropriate. □

Remark 3.5.3. The ratio $[\mathsf{O} : \mathsf{A}]$ agrees with the ratios given by the torus of trianlges (see Theorem 2.7.2), and the ratios computed in [Por94], [ES15], and [CNSS19]. The torus of triangles gives that $[\mathsf{OI} : \mathsf{AI}] = 1 : 1$, which disagrees with the sphere ratio $2 : 1$.

3.6. **Additional Properties.** What follows is a collection of additional observations about the sphere of triangles.

3.6.1. *Interpretations of the Coordinates.* We map a similarity class of triangles to a point in $\mathbb{R}^3$ of length 1. Are there geometric interpretations of the different coordinates?

Proposition 3.6.2. *Let* $p \in \mathbb{C}^3$ *be a triangle with side lengths* $a, b, c \geq 0$ *that satisfies* $a^2 + b^2 + c^2 = 1$. *Then the image of* p *on* S^2 *is*

$$\operatorname{im}(p) = (\sqrt{3}(a^2 - b^2), -4\sqrt{3}\operatorname{area}(p), 2c^2 - a^2 - b^2).$$

The statement about the area being proportional to the y-coordinate confirms a result in [CNSS19].

Proof. The result follows by computation and utilizing the formula for the area of the triangle given by

$$\operatorname{area}(\triangle ABC) = \frac{i}{4}\det\begin{bmatrix} A & \overline{A} & 1 \\ B & \overline{B} & 1 \\ C & \overline{C} & 1 \end{bmatrix}.$$

□

Corollary 3.6.3. *Let* $p = (A, B, C)$ *be a triangle with side lengths* a, b, c *satsifying* $a^2 + b^2 + c^2 = 1$*. If* $h(p)$ *satisfies*

(a) $y = 0$*, then* $\text{area}(p) = 0$*, implying that* p *is a degenerate triangle;*
(b) $x = -\sqrt{3}z$*, then* $b = c$*, and so* p *is isosceles with vertex angle at* A*;*
(c) $x = \sqrt{3}z$*, then* $a = c$*, and so* p *is isosceles with vertex angle at* B*;*
(d) $x = 0$*, then* $a = b$*, and so* p *is isosceles with vertex angle at* C*;*
(e) $z = \sqrt{3}x - 1$*, then* $b^2 + c^2 = a^2$*, and so* p *is right with right angle at* A*;*
(f) $z = -\sqrt{3}x - 1$*, then* $a^2 + c^2 = b^2$*, and so* p *is right with right angle at* B*;*
(g) $z = \frac{1}{2}$*, then* $a^2 + b^2 = c^2$*, and so* p *is right with right angle at* C*;*
(h) $z = 0$*, then* $a^2 + b^2 = 2c^2$*. We call* p *an "anti-isosceles" triangle.*

Proof. Combine the result of Proposition 3.4.1 and Proposition 3.6.2. □

3.6.4. *Antipodal Triangles.*

Proposition 3.6.5. *Let* $(A, B, C) \in \mathbb{C}^3$ *be a triangle. Then the images of the triangle* $(A, B, C) \in \mathbb{C}^3$ *and its conjugate dual triangle* $\overline{(B - C, C - A, A - B)} \in \mathbb{C}^3$ *under the hopf map* h *lie on antipodal points of the sphere.*

Proof. The result follows immediately from applying p to the triangle (A, B, C) and $\overline{(B - C, C - A, A - B)}$. □

Note that the *conjugate dual triangle* of (A, B, C) takes the directed opposite side vectors $a = B - C$, $b = C - A$, and $c = A - B$ and conjugates them.

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